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Hamiltonian technique for the construction of asymptotically flat metrics I. Stationary axisymmetric gravitational field

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Abstract. As a first step towards the investigation of more general asymptotically flat metrics a new technique is devised which, within the framework of the Hamilton formalism and using proper boundary conditions for a stationary axisymmetric gravitational field, yields the Kerr metric. This metric appears as a 'first-order correction' with respect to the Schwarzschild metric which is built into the more general metric. This goal can only be attained if one introduces so called 'kinematical momenta'. A physical interpretation of these momenta becomes possible if one studies the force exerted by the Kerr field on a spinning test particle.

1. Introduction

In a recent article Berger *et al* (1972) rederived the Schwarzschild metric by applying the ADM formalism (Arnowitt and Deser 1959, Arnowitt *et al* 1959, 1960a, b, c, d, 1961a, b, Dirac 1958a, b) to a spherically symmetric system. Unfortunately when we apply the same method to an axisymmetric system, in the hope of recovering the Kerr metric, the result is a system of two non-linear differential equations which we are unable to integrate. At this point we introduce a construction technique which is based on some reasonable assumptions and foundations in view of later applications to more general situations. Since we want to work with asymptotically flat metrics we start our discussion with the asymptotic expression for such a metric, as given by Misner *et al* (1973, p 449, to be referred to as MTW).

$$ds^2 = -\left[1-\frac{2M}{r}+O\left(\frac{1}{r^3}\right)\right]dt^2 - 2\left[2\epsilon_{jkl}\frac{J^k x^l}{r^3}+O\left(\frac{1}{r^3}\right)\right]dx^j\,dt + \left[\left(1+\frac{2M}{r}\right)\delta_{ij}+O\left(\frac{1}{r^3}\right)\right]dx^i\,dx^j \qquad (1.1)$$

where M stands for the mass and J^k for the spin of the source of the gravitational field. Radiation would cause extra terms in r^{-1} to occur in (1.1). In terms of the ADM line element

$$ds^2 = (-N^2+N_iN^i)\,dt^2 + 2N_i\,dx^i\,dt + \gamma_{ij}\,dx^i\,dx^j \qquad (1.2)$$

where N and N^i are the so called lapse and shift functions and $\|\gamma_{ij}\|$ is the metric tensor in

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three-space. We now define the following metric as a generalization of (1.1):

$$\gamma_{11} := e^{2\mu} = \frac{r^{n+2} + \delta_{n+1} r^{n+1} + \dots}{r^{n+2} + (\sigma_{n+1} - 2M) r^{n+1} + \dots} \tag{1.3a}$$

$$\gamma_{22} := e^{2\lambda} = \frac{r^{n+2} \alpha_{n+1} r^{n+1} + \dots}{r^n + \beta_{n-1} r^{n-1} + \dots} \tag{1.3b}$$

$$\gamma_{33} := e^{2\rho} \sin^2\theta = \frac{r^{n+4} + \epsilon_{n+3} r^{n+3} + \dots}{r^{n+2} + \eta_{n+1} r^{n+1} + \dots} \sin^2\theta \tag{1.3c}$$

$$N^1 = \frac{\xi_n r^n + \xi_{n-1} r^{n-1} + \dots}{r^{n+2} + \chi_{n+1} r^{n+1} + \dots} \tag{1.3d}$$

$$N^2 = \frac{\psi_{n+1} r^{n+1} + \psi_n r^n + \dots}{r^{n+4} + \zeta_{n+3} r^{n+3} + \dots} \tag{1.3e}$$

$$N^3 = \frac{\omega_{n+1} r^{n+1} + \omega_n r^n + \dots}{r^{n+4} + \kappa_{n+3} r^{n+3} + \dots} \tag{1.3f}$$

where α_i, β_i, δ_i, ... are arbitrary unknown functions of θ, ϕ and t. Their number is limited by the value of the parameter n, since we forbid the occurrence of negative indices and exponents. N can be recovered from the other metric parameters. We remark that the Schwarzschild metric is built in for $n = 0$. The most general metric is found by the substitution of (1.3) into the Einstein equations, for arbitrary large values of n. As a result one obtains an infinite system of linear differential relations. These have to be integrated by algebraic and analytic manipulations for increasing values of n, starting at $n = 0$. This technique will be demonstrated in a subsequent paper for general (i.e. non-symmetrical) time-dependent systems (for $n = 0, 1$). In that article we will also discuss the generality of the metric (1.3). As a last remark we would like to point out that different values of n yield non-identical linear differential relations between the metric functions because of the changing number of variables. It is therefore not established *a priori* that a solution of the Einstein equations for n is also the only solution for $n' > n$.

We now outline a few elements of the Hamilton formalism. In the ADM formalism one defines the action as:

$$I := \int (\pi^{ij} \dot{\gamma}_{ij} - N\mathscr{H}^0 - N_i \mathscr{H}^i) \, dt \, d^3x. \tag{1.4}$$

Here we use the ADM notation, except for $\gamma_{ij} := {}^{(3)}g_{ij\,\mathrm{ADM}}$. It is always understood in § 1 and § 2 that all quantities are defined in three-space so that we can omit the suffix (3) everywhere. In (1.4) γ_{ij}, π^{ij}, N and N_i are independent variables and variation of the action (1.4) with respect to these variables yields the Einstein equations. $\mathscr{H}^0$ and $\mathscr{H}^i$ are defined in the following way:

$$\mathscr{H}^0 := \gamma^{-1/2}[\pi^{ij}\pi_{ij} - \tfrac{1}{2}(\pi^l{}_l)^2] - \gamma^{1/2} R$$
$$\mathscr{H}^i := -2\pi^{ij}{}_{|j}. \tag{1.5}$$

Variation of the action with respect to N and N_i determines the initial value equations for π^{ij}:

$$\mathscr{H}^\mu = 0 \qquad (\mu = 0, 1, 2, 3). \tag{1.6}$$

the six dynamical Einstein equations can be obtained by varying I with respect to the γ_{ij}. Using relation (1.5) one then finds:

$$\dot{\pi}^{ij} = N\gamma^{1/2}(\gamma^{ij}R - R^{ij}) - 2N\gamma^{-1/2}(\pi^{i}{}_{m}\pi^{mj} - \tfrac{1}{2}\pi^{m}{}_{m}\pi^{ij})$$
$$- N^{|l}{}_{|l}\gamma^{1/2}\gamma^{ij} + \gamma^{1/2}N^{|ij} + (\pi^{ij}N^{l})_{|l} - N^{i}{}_{|m}\pi^{mj} - N^{j}{}_{|m}\pi^{mi}. \tag{1.7}$$

By varying I with respect to the π^{ij} one simply recovers the definition for the π_{ij} as:

$$\dot{\gamma}_{ij} = 2N\gamma^{-1/2}(\pi_{ij} - \tfrac{1}{2}\gamma_{ij}\pi^{l}{}_{l}) + N_{i|j} + N_{j|i}. \tag{1.8}$$

We still dispose of four coordinate conditions, three of which we use to diagonalize $\|\gamma_{ij}\|$, while the fourth one is needed to secure the slicing of the space–time by the following condition:

$$\mathrm{Tr}\,\pi = \pi^{l}{}_{l} := \pi = 0. \tag{1.9a}$$

This condition is valid on an arbitrary initial space-like hypersurface. Using the dynamical equations (1.7) one can show that (1.9*a*) induces an equation for the lapse function N:

$$N^{|l}{}_{|l} = NR. \tag{1.9b}$$

2. Calculation of the metric in the axisymmetric case

Until now we did not require the three-metric to exhibit any particular symmetry. We define now the three-line element for an axisymmetric system by:

$$dl^2 := e^{2\mu}\,dr^2 + e^{2\lambda}\,d\theta^2 + e^{2\rho}\sin^2\theta\,d\phi^2 \tag{2.1}$$

where μ, λ and ρ are solely functions of r, θ and t. In order to have the correct asymptotic form the metric must agree with the general form given by MTW, i.e.:

$$\gamma_{11} = \left[1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right)\right]^{-1}$$
$$\gamma_{22} = \gamma_{33}\sin^{-2}\theta = [r^2 + O(r^0)]$$
$$(N)^2 = 1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right)$$
$$N_3 = -\frac{2Mar\sin^2\theta}{r^2} + O\left(\frac{1}{r^3}\right) \tag{2.2}$$

where $a := J/M$, J being the angular momentum of the source. The Lagrange multipliers N and N_i vanish in this coordinate system. The momenta can be defined by:

$$\|\pi^{ij}\| = \begin{pmatrix} \tfrac{1}{2}\pi_\mu\,e^{-2\mu} & \pi^{12} & \pi^{13} \\ \pi^{21} & \tfrac{1}{2}\pi_\lambda\,e^{-2\lambda} & \pi^{23} \\ \pi^{31} & \pi^{32} & \tfrac{1}{2}\pi_\rho\,e^{-2\rho}\sin^{-2}\theta \end{pmatrix}. \tag{2.3}$$

We remark that it is not inconsistent to have off-diagonal π_{ij} appearing in (2.3), because they may result from the initial value equations which we have not written down yet. As we can read from the action:

$$I=\int dt\, d^3x(\pi_\mu\dot{\mu}+\pi_\lambda\dot{\lambda}+\pi_\rho\dot{\rho}-N\mathscr{H}^0-N_i\mathscr{H}^i) \tag{2.4}$$

the diagonal elements $\{\pi_\mu, \pi_\lambda, \pi_\rho\}$ are the momenta conjugated to the quantities $\{\mu, \lambda, \rho\}$ and they completely determine the dynamics of the system. We note finally that the slicing condition (1.8*a*, 1.8*b*) reduces the number of degrees of freedom from three, in equation (2.3), to two, as could be expected. As we mentioned in the introduction we will look for more general metrics in a further article; at the present time however, we limit ourselves to the search for a stationary metric; thus

$$\dot{\gamma}_{ij}=0. \tag{2.5}$$

In these circumstances the equations (1.7) assign to the momenta the following values:

$$\pi_\mu=\pi_\lambda=\pi_\rho=0 \tag{2.6a, b, c}$$

which already eliminates all dynamical freedom and

$$\pi_{12}=0 \tag{2.6d}$$

$$\pi_{13}=\frac{e^{\mu+\lambda+3\rho}}{2N}\sin^3\theta(N^3)' \tag{2.6e}$$

$$\pi_{23}=-\frac{e^{\mu+\lambda+3\rho}}{2N}\sin^3\theta(N^3)^{;}. \tag{2.6f}$$

Differentiation with respect to t, r and θ is denoted respectively by a dot, a prime and a semicolon. As a next step we calculate $\mathscr{H}^0$ and $\mathscr{H}^i$ and find:

$$\mathscr{H}^1\equiv\mathscr{H}^2\equiv 0 \tag{2.7a, b}$$

$$\mathscr{H}^3:=-2(\pi^{13})'-2(\pi^{23})^{;}-4\rho'\pi^{13}-4(\rho^{;}+\cot\theta)\pi^{23} \tag{2.7c}$$

$$\begin{aligned}\mathscr{H}^0=e^{-\mu-\lambda-\rho}\sin^{-1}\theta\{&2\pi_{23}\pi^{23}+2\pi_{31}\pi^{31}+2\,e^{2(\lambda+\rho)}\\&\times\sin^2\theta[\lambda''+\rho''-\mu'\lambda'-\mu'\rho'+\rho'\lambda'+(\lambda')^2+(\rho')^2]+2\,e^{2(\mu+\rho)}\\&\times\sin^2\theta[\mu^{;;}+\rho^{;;}+\mu^{;}\rho^{;}+(\mu^{;})^2+(\rho^{;})^2-\tfrac{1}{2}\lambda^{;}(\mu^{;}+\rho^{;})-1\\&+\cot\theta(\mu^{;}+\rho^{;}-\tfrac{1}{2}\lambda^{;})]\}.\end{aligned} \tag{2.7d}$$

Since $\mathscr{H}^3$ involves only non-dynamical variables, the constraint equation $\mathscr{H}^3=0$ must follow as a result of the other Einstein equations. The four initial value equations read:

$$\mathscr{H}^1\equiv\mathscr{H}^2\equiv 0 \tag{2.8a, b}$$

$$\mathscr{H}^3=0 \tag{2.8c}$$

$$\mathscr{H}^0=0 \tag{2.8d}$$

with (2.8*c*) a consequence of (2.8*d*) and the dynamic equations. When we replace the value of R by $N^{|l}{}_{|l}/N$ in (2.8*d*) we include the slicing condition (1.9*b*). By the same token we guarantee that the dynamical equations for $\dot{\pi}_\mu$, $\dot{\pi}_\lambda$ and $\dot{\pi}_\rho$ are satisfied. Only one dynamical equation is left, that for $\dot{\pi}^{12}$. Those for $\dot{\pi}^{13}$ and $\dot{\pi}^{23}$ are identically zero. Thus:

$$N[\rho^{;\prime}-\mu^{;}\rho'-(\rho^{;}+\cot\theta)(\lambda'-\rho')]+N^{;\prime}-\mu^{;}N'-\lambda'N^{;}-\frac{e^{2\rho}}{2N}(N^3)'(N^3)^{;}\sin^2\theta=0. \tag{2.9}$$

(2.8*d*) combined with (1.9*b*) and (2.9) are the Einstein equations which remain to be solved. Their solution is not straightforward. By deducing a functional form for the multiplier N we can bring in a small simplification. We find N by comparing the three- and four-volume densities $(\gamma)^{1/2}$ and $(-{}^{(4)}g)^{1/2}$ for the spherically symmetric (SS) and axisymmetric (AS) case:

$$(-{}^{(4)}g_{SS})^{1/2}=e^{2\lambda_{SS}}\sin\theta \qquad (\gamma_{SS})^{1/2}=e^{\mu_{SS}+2\lambda_{SS}}\sin\theta$$

$$N_{SS}=e^{-\mu_{SS}} \tag{2.10}$$

where $\lambda_{SS}=\rho_{SS}$ and

$$(-{}^{(4)}g)^{1/2}=N(\gamma)^{1/2}$$

is the binding relation between the three quantities. In the axisymmetric case $(\gamma_{AS})^{1/2}$ is given by:

$$(\gamma_{AS})^{1/2}=e^{\mu_{AS}+\lambda_{AS}+\rho_{AS}}\sin\theta.$$

This still leaves some freedom in the choice of N and $(-{}^{(4)}g)^{1/2}$. For purely phenomenological reasons we choose

$$(-{}^{(4)}g_{AS})^{1/2}=e^{2\lambda_{AS}}\sin\theta$$

and

$$N_{AS}=e^{-\mu_{AS}+\lambda_{AS}-\rho_{AS}}$$

so that for J tending to zero the axisymmetric solution will smoothly turn into a spherically symmetric metric.

Finally we introduce the special form (1.3) for the metric into the Einstein equations and we get for $n=0$: as *ansatz*

$$\gamma_{22}:=e^{2\lambda}=r^2+\beta_1 \tag{2.11a}$$

$$\gamma_{33}:=e^{2\rho}\sin^2\theta=\frac{r^4+d_2r^2+d_1r+d_0}{r^2+\beta_2}\sin^2\theta \tag{2.11b}$$

$$\gamma_{11}:=e^{2\mu}=\frac{r^2+\beta_3}{r^2-2Mr+d_3} \tag{2.11c}$$

$$N=e^{-\mu+\lambda-\rho} \tag{2.11d}$$

$$N_1=0 \tag{2.11e}$$

$$N_2=0 \tag{2.11f}$$

$$N_3=\frac{2Mar\sin^2\theta}{r^2+\beta_4}. \tag{2.11g}$$

We notice that the role of the singularity $r=0$ of the SS case has been taken over by $r^2+\beta_i=0$ in the AS case. Since we want the physical interpretation of both singularities to be the same—as the location of the source of the gravitational field—it is easy to see all β_i have to be equal. Thus:

$$\beta_i=\beta. \tag{2.11h}$$

Now in each of the Einstein equations we can group all terms on a common denominator as coefficients of a certain power of r. In order to satisfy the equations, each of the coefficients has to vanish. This leads to a sizable number of sub-equations. Only a small fraction of them is linear in the metric parameters. We will thus integrate the linear equations and check the solution on the non-linear ones. When we rescale $r^{n_{max}}$ to be r^0 we find for $\mathscr{H}^0$ the following results: the coefficients of r^0 and r^{-1} are identically zero; this is something we would expect, because we had the Schwarzschild metric built into (1.3). The vanishing of the coefficient of r^{-2} then gives:

$$d_3^{\cdot\cdot}+d_3^{\cdot}\cot\theta+2d_3-d_2^{\cdot\cdot}-d_2^{\cdot}\cot\theta-2d_2+\beta^{\cdot\cdot}+\beta^{\cdot}\cot\theta+2\beta=0. \tag{2.12a}$$

The same application to the coefficient of order r^{-3} leads to:

$$\begin{aligned}-2Md_3^{\cdot\cdot}-2Md_3^{\cdot}\cot\theta-8Md_3+4Md_2^{\cdot\cdot}+4Md_2^{\cdot}\cot\theta+16Md_2\\ -d_1^{\cdot\cdot}-d_1^{\cdot}\cot\theta-6d_1-4M\beta^{\cdot\cdot}-4M\beta^{\cdot}\cot\theta-16M\beta=0.\end{aligned} \tag{2.12b}$$

It follows from (2.12*b*) that d_1 contains M linearly. Therefore in order r^{-4} we get two equations instead of one:

(*a*) Order M^0:

$$\begin{aligned}&d_3^{\cdot\cdot}(2\beta+d_3+2d_2)+d_3^{\cdot}(-d_3^{\cdot}+d_3\cot\theta+2\beta\cot\theta+2d_2\cot\theta)\\ &\quad+d_2^{\cdot\cdot}(-d_2-2d_3-2\beta)+d_2^{\cdot}(d_2^{\cdot}-d_2\cot\theta-2\beta\cot\theta-2d_3\cot\theta)\\ &\quad+\beta^{\cdot\cdot}(2d_3+2d_2+\beta)+\beta^{\cdot}(-\beta^{\cdot}+2d_2\cot\theta+\beta\cot\theta+2d_3\cot\theta)\\ &\quad-d_0^{\cdot\cdot}-d_0^{\cdot}\cot\theta-12d_0+2(d_2)^2-6d_2d_3+4(d_3)^2+14\beta d_3\\ &\quad-2\beta^2=0\end{aligned} \tag{2.12c_1}$$

(*b*) *order* M^2:

$$\begin{aligned}&-4M^2d_2^{\cdot\cdot}-4M^2d_2^{\cdot}\cot\theta+4Md_1^{\cdot\cdot}+4Md_1^{\cdot}\cot\theta+4M\beta^{\cdot\cdot}+4M^2\beta^{\cdot}\\ &\quad\times\cot\theta+42Md_1-40M^2d_2+8M^2d_3+40M^2\beta-36M^2a^2\sin^2\theta=0.\end{aligned} \tag{2.12c_2}$$

From these equations we can easily deduce that

$$d_0\sim a^4 \qquad d_1\sim Ma^2 \qquad d_2, d_3, \beta\sim a^2. \tag{2.12d}$$

We can operate in a similar way for the second Einstein equation. Again the coefficients of order r^0 and r^{-1} yield zero. The next order, r^{-2}, gives the following equation:

$$2d_2^{\cdot}-4\beta^{\cdot}-2d_3^{\cdot}+\cot\theta(-4d_2+8\beta)=0. \tag{2.13a}$$

Order r^{-3} gives:

$$-6Md_2^{\cdot}+2d_1^{\cdot}+8M\beta^{\cdot}+\cot\theta(-6d_1+8Md_2-16M\beta)=0. \tag{2.13b}$$

In order r^{-4} we can again distinguish between order M^0:

$$\begin{aligned}&d_2^{\cdot}(4d_2+8\beta)+d_3^{\cdot}(2d_2)+\beta^{\cdot}(-8d_2-4d_3-8\beta)+2d_0^{\cdot}\\ &\quad+\cot\theta[-8d_0-8(d_2)^2+8\beta^2-4d_2d_3+16d_2\beta+8d_3\beta]=0\end{aligned} \tag{2.13c_1}$$

and order M^2:

$$-6Md_1^{\cdot}+12Md_1\cot\theta=0. \tag{2.13c_2}$$

The result of the integration is:

$$d_1=2Ma^2\sin^2\theta \tag{2.14a}$$

$$d_2 = a^2 + a^2 \cos^2\theta \tag{2.14b}$$

$$\beta = a^2 \cos^2\theta \tag{2.14c}$$

$$d_3 = a^2 \tag{2.14d}$$

$$d_0 = a^4 - a^4 \sin^2\theta. \tag{2.14e}$$

We still have to check all nonlinear equations and we find that they vanish when we insert (2.14) in them, so that the metric with parameters (2.14) is an exact solution for the Einstein equations, the Kerr metric:

$$\gamma_{11} := e^{2\mu} = \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2} \tag{2.15a}$$

$$\gamma_{22} := e^{2\lambda} = r^2 + a^2\cos^2\theta \tag{2.15b}$$

$$\gamma_{33} := e^{2\rho}\sin^2\theta = \frac{(r^2+a^2)^2 - a^2\sin^2\theta(r^2-2Mr+a^2)}{r^2+a^2\cos^2\theta} \tag{2.15c}$$

$$N_3 = -\frac{2Mar\sin^2\theta}{r^2+a^2\cos^2\theta} \tag{2.15d}$$

$$N_1 = N_2 = 0 \tag{2.15e, f}$$

$$(N)^2 = e^{-2\mu+2\lambda-2\rho} = \frac{(r^2-2Mr+a^2)(r^2+a^2\cos^2\theta)}{(r^2+a^2)^2 - a^2\sin^2\theta(r^2-2Mr+a^2)}. \tag{2.15g}$$

The off-diagonal momenta can now also be calculated:

$$\pi^{13} = \frac{Ma\sin\theta[r^2(3r^2-a^2) - a^2\cos^2\theta(r^2-a^2)]}{(r^2+a^2\cos^2\theta)[(r^2+a^2)^2 - a^2\sin^2\theta(r^2-2Mr+a^2)]} \tag{2.15h}$$

$$\pi^{23} = \frac{2Ma^3 r\sin^2\theta\cos\theta}{(r^2+a^2\cos^2\theta)[(r^2+a^2)^2 - a^2\sin^2\theta(r^2-2Mr+a^2)]}. \tag{2.15i}$$

We propose to call these off-diagonal momenta 'kinematical momenta' so that the distinction between these momenta and dynamical ones on the diagonal becomes clearer.

3. Physical interpretation of the kinematical momenta

A detailed study of a spinning test particle in an asymptotically flat space–time has been made before (Schiff 1960, Wald 1972, Wilkins 1970), so we shall confine ourselves to a mere outline of the theory on this topic. Throughout this section we will use Wald's notation except that four-space–time quantities will now have a (4) suffix. The equation of motion for a spinning test particle is given by:

$$\frac{D^{(4)}p^\mu}{Ds} = -\tfrac{1}{2}{}^{(4)}R^\mu{}_{\nu\rho\sigma}\,{}^{(4)}v^{\nu}\,{}^{(4)}S^{\rho\sigma} \tag{3.1}$$

where s is the arc length, ${}^{(4)}p^\mu$ the four-momentum, ${}^{(4)}v^\nu$ the four-velocity and ${}^{(4)}S^{\rho\sigma}$ the spin-tensor which obeys the equation:

$$\frac{D^{(4)}S^{\mu\nu}}{Ds}=2^{(4)}p^{[\mu(4)}v^{\nu]}=-\frac{1}{2M}(-^{(4)}g)^{1/2(4)}\epsilon^{\mu\nu\lambda\rho(4)}R_{\lambda\alpha\beta\gamma}{}^{(4)}v^{\alpha(4)}S^{\beta\gamma(4)}S_\rho \tag{3.2}$$

for the torsion. ${}^{(4)}S_\rho$ is the spin vector which is related to the spin-tensor in the following way:

$${}^{(4)}S_\rho=\frac{1}{2M}(-^{(4)}g)^{1/2(4)}\epsilon_{\mu\nu\lambda\rho}{}^{(4)}p^{\mu\,(4)}S^{\nu\lambda}. \tag{3.3}$$

The equation which forces the particle on a centre-of-mass path is

$${}^{(4)}p_\mu{}^{(4)}S^{\mu\nu}=0. \tag{3.4}$$

M is the mass of the test particle:

$$M^2={}^{(4)}p_\mu{}^{(4)}p^\mu, \tag{3.5}$$

while the spin S is given by:

$$S^2=\tfrac{1}{2}{}^{(4)}S_{\mu\nu}{}^{(4)}S^{\mu\nu}. \tag{3.6}$$

The spin of a test particle is limited by the relation:

$$S/M\leqslant r_0 \tag{3.7}$$

where r_0 is the dimension of the particle, so that the outer surface of the particle does not rotate faster than the speed of light. It is then possible to calculate (3.1) in first order for the metric (1.1) explicitly. Working in the isotropic form of (1.1) Wald finds that since the test particle is initially at rest:

$${}^{(4)}v^\mu\simeq(1,0,0,0)$$

and

$${}^{(4)}p_\mu{}^{(4)}S^{\mu\nu}\simeq M^{(4)}v_\mu{}^{(4)}S^{\mu\nu}=0$$

and therefore:

$$S^{0i}=-S^{i0}=0 \qquad S_{jk}=\epsilon_{jkl}S^l \tag{3.8}$$

the equation for the generalized force (3.1) can be reduced to:

$$F^i_{\mathrm{G}}=-\tfrac{1}{2}{}^{(4)}R^i{}_{0jk}\epsilon^{jkl}S_l. \tag{3.9}$$

This becomes in first order:

$$\boldsymbol{F}_{\mathrm{G}}=-\nabla\left(\frac{-\boldsymbol{S}\cdot\boldsymbol{J}+3(\boldsymbol{S}\cdot\hat{\boldsymbol{r}})(\boldsymbol{J}\cdot\hat{\boldsymbol{r}})}{r^3}\right)+O\left(\frac{1}{r^5}\right). \tag{3.10a}$$

(3.10a) demonstrates the similarity, up to the sign, with the force term describing the interaction between two magnetic dipoles:

$$\boldsymbol{F}_{\mathrm{M}}=-\nabla\left(\frac{\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2-3(\boldsymbol{\mu}_1\cdot\hat{\boldsymbol{r}})(\boldsymbol{\mu}_2\cdot\hat{\boldsymbol{r}})}{r^3}\right). \tag{3.10b}$$

A different use can be made of equation (3.9) by applying the Codazzi equation to it (Misner *et al* 1973, p 514):

$${}^{(4)}R^0{}_{ijk}=(K_{ij|k}-K_{ik|j}) \tag{3.11}$$

where K_{ij} is the second fundamental tensor defined by

$$K_{ij} = -\tfrac{1}{2}\pounds_n \gamma_{ij} \tag{3.12}$$

n being the normal to a space-like hypersurface. K_{ij} is related to π_{ij} in the following way:

$$\pi^{ij} = \gamma^{1/2}(\gamma^{ij}K^m{}_m - K^{ij}). \tag{3.13}$$

Using (3.11) in (3.9) we get:

$$F_{Gi} \simeq \epsilon^{jkl} S_l K_{ij|k}. \tag{3.14}$$

If we define β_i^l to be the negative curl of the $\boldsymbol{K}$ field we can write

$$\boldsymbol{F}_G \simeq \boldsymbol{S} \cdot \boldsymbol{\beta}. \tag{3.15a}$$

This is similar to the expression for the magnetic force (Jackson 1962).

$$\boldsymbol{F}_M = (\boldsymbol{\mu} \cdot \boldsymbol{\nabla})\boldsymbol{B} \tag{3.15b}$$

where $\boldsymbol{B}$ is the magnetic field. Thus through equations (3.13) and (3.15a), the kinematical momenta determine the force on a spinning test particle much in the same way as the magnetic field causes two magnetic dipoles to interact according to (3.15b).

4. Conclusion

The construction technique (1.3) applied on a stationary axisymmetric system was not sufficient to arrive at the Kerr solution. A few extra assumptions had to be made in order to get this result. For the discussion of a non-stationary metric—which goes asymptotically like the Kerr metric—these same assumptions will still have to be valid. Only then will the usefulness of the entire method be apparent.

One point which this technique has revealed is the similarity that exists between the $\boldsymbol{K}$ field (consisting entirely of kinematical momenta, for the gravitational field in the case of the Kerr metric) and the magnetic field $\boldsymbol{B}$ induced by a magnetic dipole.

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